Formalization of localization problem: Discrete Bayesian filters

Olivier Aycard

Professor

Grenoble INP - PHELMA

GIPSA Lab

https://www.gipsa-lab.grenoble-inp.fr/user/olivier.aycard

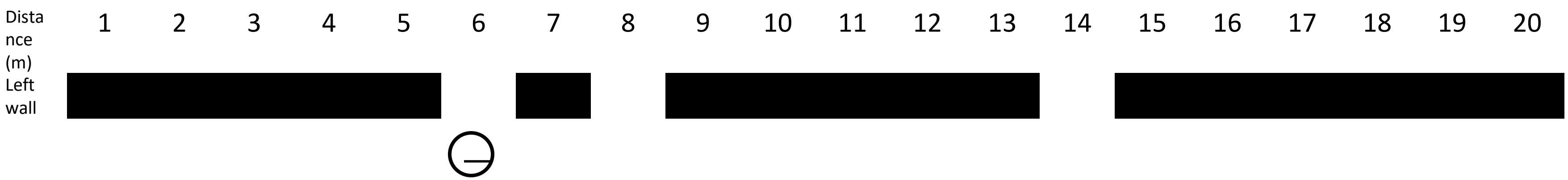
olivier.aycard@grenoble-inp.fr

The localization problem

- Localization of a mobile robot inside a corridor: 1D localization problem

1. Initial position known
    1. Using actions to predict position
        1. Motions are perfect: one example
        2. Motions are not perfect: one example + formalization
        3. Complete formalization: dynamic model + prediction
    2. Using observations to estimate position
        1. Observations are pefect: one example
        2. Observations are not perfect: one example + formalization
        3. Observations are not perfect: second example
        4. Complete formalization: sensor model + estimation
2. Initial position unknown
3. Algorithm to perform localization
4. Real examples

The localization problem



- Initial position known, motions are perfect

1. $P(S_0 = 6) = 1$
2. the mobile robot moves from 1 meter ($A_1 = 1$)

   $P(S_1 = 7 \mid A_1 = 1) = 1$
3. the mobile robot moves again from 1 meter ($A_2 = 1$)

   $P(S_2 = 8 \mid A_2 = 1, A_1 = 1) = 1$
4. the mobile robot moves again from 1 meter ($A_3 = 1$)

   $P(S_3 = 9 \mid A_3 = 1, A_2 = 1, A_1 = 1) = 1$
5. the mobile robot moves again from 1 meter ($A_4 = 1$)

   $P(S_4 = 10 \mid A_4 = 1, A_3 = 1, A_2 = 1, A_1 = 1) = 1$

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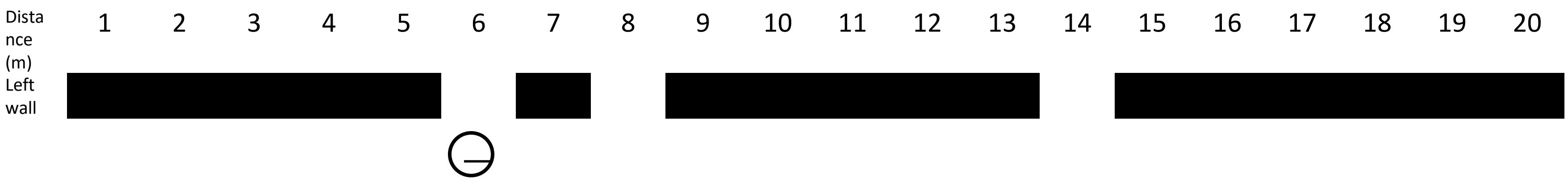
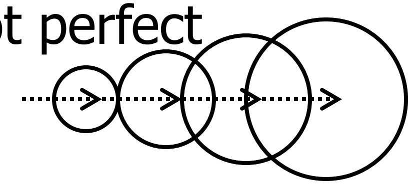


$P(S_0 = 6) = 1$

- But motions are not perfect



The robot is lost !!!

- When the mobile robot moves from 1 meter, the mobile robot is located:
  - 1 meter further with a probability of 80%
  - at the same location with a probability of 10%
  - 2 meters further with a probability of 10%

- After one action:

$P(S_1 = 6 \mid A_1 = 1) = 0.1 \times 1 = 0.1$

$P(S_1 = 7 \mid A_1 = 1) = 0.8 \times 1 = 0.8$

$P(S_1 = 8 \mid A_1 = 1) = 0.1 \times 1 = 0.1$

The localization problem

- After one action:

$P(S_1 = 6 \mid A_1 = 1) = 0.1 \times 1 = 0.1$
$P(S_1 = 7 \mid A_1 = 1) = 0.8 \times 1 = 0.8$
$P(S_1 = 8 \mid A_1 = 1) = 0.1 \times 1 = 0.1$

- After one action (formalization):

$P(S_1 = 6 \mid A_1 = 1) = P(S_1 = 6 \mid A_1 = 1, S_0 = 6) \times P(S_0 = 6)$
$P(S_1 = 7 \mid A_1 = 1) = P(S_1 = 7 \mid A_1 = 1, S_0 = 6) \times P(S_0 = 6)$
$P(S_1 = 8 \mid A_1 = 1) = P(S_1 = 8 \mid A_1 = 1, S_0 = 6) \times P(S_0 = 6)$

- For s = {6, 7, 8}
  $P(S_1 = s \mid A_1 = 1) = P(S_1 = s \mid A_1 = 1, S_0 = 6) \times P(S_0 = 6)$

The localization problem

- After two actions:

$P(S_2 = 6 \mid A_1 = 1, A_2 = 1) = 0.1 \times 0.1 = 0.01$
$P(S_2 = 7 \mid A_1 = 1, A_2 = 1) = 0.1 \times 0.8 + 0.8 \times 0.1 = 0.16$
$P(S_2 = 8 \mid A_1 = 1, A_2 = 1) = 0.1 \times 0.1 + 0.8 \times 0.8 + 0.1 \times 0.1 = 0.66$
$P(S_2 = 9 \mid A_1 = 1, A_2 = 1) = 0.8 \times 0.1 + 0.1 \times 0.8 = 0.16$
$P(S_2 = 10 \mid A_1 = 1, A_2 = 1) = 0.1 \times 0.1 = 0.01$

- After two actions (formalization):
- $P(S_2 = 6 \mid A_1 = 1, A_2 = 1) = P(S_2 = 6 \mid A_2 = 1, S_1 = 6) \times P(S_1 = 6 \mid A_1 = 1)$
- $P(S_2 = 7 \mid A_1 = 1, A_2 = 1) = P(S_2 = 7 \mid A_2 = 1, S_1 = 7) \times P(S_1 = 7 \mid A_1 = 1)$
  $+ P(S_2 = 7 \mid A_2 = 1, S_1 = 6) \times P(S_1 = 6 \mid A_1 = 1)$
- $P(S_2 = 8 \mid A_1 = 1, A_2 = 1) = P(S_2 = 8 \mid A_2 = 1, S_1 = 8) \times P(S_1 = 8 \mid A_1 = 1)$
  $+ P(S_2 = 8 \mid A_2 = 1, S_1 = 7) \times P(S_1 = 7 \mid A_1 = 1)$
  $+ P(S_2 = 8 \mid A_2 = 1, S_1 = 6) \times P(S_1 = 6 \mid A_1 = 1)$

…

For s = {6, …, 10} and s’ = {6, 7, 8}
$P(S_2 = s \mid A_1 = 1, A_2 = 1) = \sum_{s'} P(S_2 = s \mid A_2 = 1, S_1 = s') \times P(S_1 = s' \mid A_1 = 1)$

The localization problem

- After three actions:

$P(S_3 = 6 \mid A_1 = 1, A_2 = 1, A_3 = 1) = 0.01 \times 0.1 = 0.001$

$P(S_3 = 7 \mid A_1 = 1, A_2 = 1, A_3 = 1) = 0.01 \times 0.8 + 0.16 \times 0.1 = 0.024$

$P(S_3 = 8 \mid A_1 = 1, A_2 = 1, A_3 = 1) = 0.01 \times 0.1 + 0.16 \times 0.8 + 0.66 \times 0.1 = 0.195$

$P(S_3 = 9 \mid A_1 = 1, A_2 = 1, A_3 = 1) = 0.16 \times 0.1 + 0.66 \times 0.8 + 0.16 \times 0.1 = 0.56$

$P(S_3 = 10 \mid A_1 = 1, A_2 = 1, A_3 = 1) = 0.66 \times 0.1 + 0.16 \times 0.8 + 0.01 \times 0.1 = 0.195$

$P(S_3 = 11 \mid A_1 = 1, A_2 = 1, A_3 = 1) = 0.16 \times 0.01 + 0.01 \times 0.8 = 0.024$

$P(S_3 = 12 \mid A_1 = 1, A_2 = 1, A_3 = 1) = 0.01 \times 0.1 = 0.001$

- For $s = \{6, \ldots, 12\}$ and $s' = \{6, \ldots, 10\}$
  $P(S_3 = s \mid A_1 = 1, A_2 = 1, A_3 = 1) =$
  $\sum_{s'} P(S_3 = s \mid A_3 = 1, S_2 = s') \times P(S_2 = s' \mid A_1 = 1, A_2 = 1)$

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The localization problem (formalization)

- When the mobile robot moves from 1 meter, the mobile robot is located:
  - 1 meter further with a probability of 80%
  - at the same location with a probability of 10%
  - 2 meters further with a probability of 10%

$P(S_{t+1} = s+1 \mid A_{t+1} = 1, S_t = s) = 0.8$
$P(S_{t+1} = s \mid A_{t+1} = 1, S_t = s) = 0.1$
$P(S_{t+1} = s+2 \mid A_{t+1} = 1, S_t = s) = 0.1$

- This model is called the dynamic model or the action model

- We use this model to predict where the mobile robot will be after doing a given action:

For each s and s′ in S
$P(S_{t+1} = s \mid A_1, \ldots, A_{t+1}) = \sum_{s'} P(S_{t+1} = s \mid A_{t+1}, S_t = s') \times P(S_t = s' \mid A_1, \ldots, A_t)$

- Recursive computation

The localization problem

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The localization problem

$P(S_1 = 6 \mid A_1 = 1) = 0.1$

$P(S_1 = 7 \mid A_1 = 1) = 0.8$

$P(S_1 = 8 \mid A_1 = 1) = 0.1$

- The mobile robot is equipped with sensors
- If sensors are perfect:
    - when the robot is in front of a wall, it will perceive a wall;
    - when the robot is in front of a door, it will perceive a door.
- Suppose that after its first action ($A_1 = 1$), it perceives a wall ($O_1 = w$):

$P(S_1 = 6 \mid A_1 = 1, O_1 = w) = 0.1 \times 0 = 0 \qquad = 0$

$P(S_1 = 7 \mid A_1 = 1, O_1 = w) = 0.8 \times 1 = 0.8 \qquad = 1$

$P(S_1 = 8 \mid A_1 = 1, O_1 = w) = 0.1 \times 0 = 0 \qquad = 0$

The localization problem

$P(S_1 = 7 \mid A_1 = 1, O_1 = w) = 1$

- The robot moves again from 1 meter:

$P(S_2 = 7 \mid A_1 = 1, O_1 = w, A_2 = 1) = 0.1 \times 1$

$P(S_2 = 8 \mid A_1 = 1, O_1 = w, A_2 = 1) = 0.8 \times 1$

$P(S_2 = 9 \mid A_1 = 1, O_1 = w, A_2 = 1) = 0.1 \times 1$

- Suppose that after its second action ($A_2 = 1$), it perceives a door ($O_2 = d$):

$P(S_2 = 7 \mid A_1 = 1, O_1 = w, A_2 = 1, O_2 = d) = 0.1 \times 0 = 0 \qquad = 0$

$P(S_2 = 8 \mid A_1 = 1, O_1 = w, A_2 = 1, O_2 = d) = 0.8 \times 1 = 0.8 \qquad = 1$

$P(S_2 = 9 \mid A_1 = 1, O_1 = w, A_2 = 1, O_2 = d) = 0.1 \times 0 = 0 \qquad = 0$

- O_1 and O_2 confirm motions

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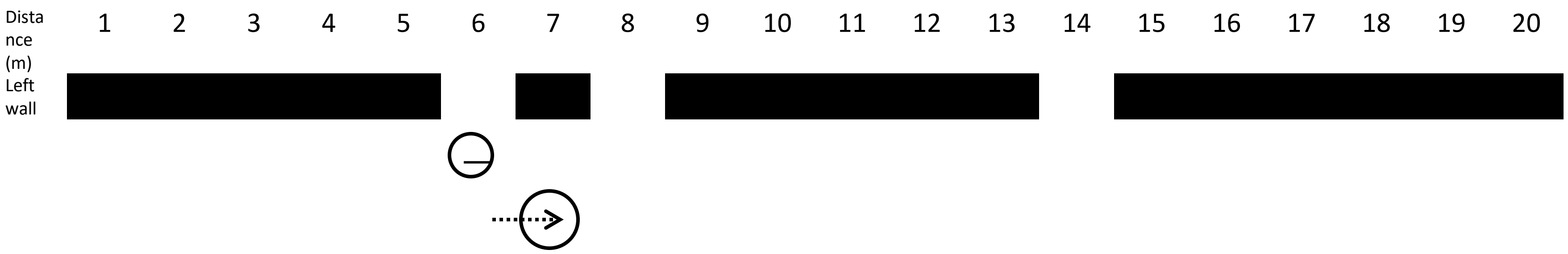

- But the sensors are not perfect
- When the mobile robot is located in front of a door, it perceives:
    - a door with a probability of 80%
    - a wall with a probability of 20%
- When the mobile robot is located in front of a wall, it perceives:
    - a wall with a probability of 90%
    - a door with a probability of 10%

The localization problem

$P(S_1 = 6 \mid A_1 = 1) = 0.1$
$P(S_1 = 7 \mid A_1 = 1) = 0.8$
$P(S_1 = 8 \mid A_1 = 1) = 0.1$

- Suppose that after its first action ($A_1 = 1$), it perceives a wall ($O_1 = w$):
  $P(S_1 = 6 \mid A_1 = 1, O_1 = w) = 0.1 \times 0.2 = 0.02 = 1/38 \quad = 0.03$
  $P(S_1 = 7 \mid A_1 = 1, O_1 = w) = 0.8 \times 0.9 = 0.72 = 18/19 \quad = 0.94$
  $P(S_1 = 8 \mid A_1 = 1, O_1 = w) = 0.1 \times 0.2 = 0.02 = 1/38 \quad = 0.03$
- Suppose that after its first action ($A_1 = 1$), it perceives a wall ($O_1 = w$):
  $P(S_1 = 6 \mid A_1 = 1, O_1 = w) = P(S_1 = 6 \mid A_1 = 1) \times P(O_1 = w \mid S_1 = 6)$
  $P(S_1 = 7 \mid A_1 = 1, O_1 = w) = P(S_1 = 7 \mid A_1 = 1) \times P(O_1 = w \mid S_1 = 7)$
  $P(S_1 = 8 \mid A_1 = 1, O_1 = w) = P(S_1 = 8 \mid A_1 = 1) \times P(O_1 = w \mid S_1 = 8)$
- O_1"confirms" A_1

The localization problem

- After two actions:

$P(S_2 = 6 \mid A_1 = 1, O_1 = w, A_2 = 1) = 0.03 \times 0.1 = 0.003$

$P(S_2 = 7 \mid A_1 = 1, O_1 = w, A_2 = 1) = 0.03 \times 0.8 + 0.94 \times 0.1 = 0.118$

$P(S_2 = 8 \mid A_1 = 1, O_1 = w, A_2 = 1) = 0.03 \times 0.1 + 0.94 \times 0.8 + 0.03 \times 0.1 = 0.758$

$P(S_2 = 9 \mid A_1 = 1, O_1 = w, A_2 = 1) = 0.94 \times 0.1 + 0.03 \times 0.8 = 0.118$

$P(S_2 = 10 \mid A_1 = 1, O_1 = w, A_2 = 1) = 0.03 \times 0.1 = 0.003$

- Suppose that after its second action ($A_2 = 1$), it perceives a wall ($O_2 = d$):

$P(S_2 = 6 \mid A_1 = 1, O_1 = w, A_2 = 1, O_2 = d) = 0.003 \times 0.8 = 0.002 = 0{,}005$

$P(S_2 = 7 \mid A_1 = 1, O_1 = w, A_2 = 1 , O_2 = d) = 0.118 \times 0.1 = 0.02 = 0{,}02$

$P(S_2 = 8 \mid A_1 = 1, O_1 = w, A_2 = 1 , O_2 = d) = 0.758 \times 0.8 = 0.6 = 0{,}95$

$P(S_2 = 9 \mid A_1 = 1, O_1 = w, A_2 = 1 , O_2 = d) = 0.118 \times 0.1 = 0.002 = 0{,}02$

$P(S_2 = 10 \mid A_1 = 1, O_1 = w, A_2 = 1 , O_2 = d) = 0.003 \times 0.8 = 0.02 = 0{,}005$

- O_2"confirms" A_2

For $s = \{6, \ldots, 10\}$

$P(S_2 = s \mid A_1 = 1, O_1 = w, A_2 = 1, O_2 = d)$

$= P(S_2 = s \mid A_1 = 1 , O_1 = w, A_2 = 1) \times P(O_2 = d \mid S_2 = s)$

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- After two actions:

$P(S_2 = 6 \mid A_1 = 1, O_1 = w, A_2 = 1) = 0.03 \times 0.1 = 0.003$

$P(S_2 = 7 \mid A_1 = 1, O_1 = w, A_2 = 1) = 0.03 \times 0.8 + 0.94 \times 0.1 = 0.118$

$P(S_2 = 8 \mid A_1 = 1, O_1 = w, A_2 = 1) = 0.03 \times 0.1 + 0.94 \times 0.8 + 0.03 \times 0.1 = 0.758$

$P(S_2 = 9 \mid A_1 = 1, O_1 = w, A_2 = 1) = 0.94 \times 0.1 + 0.03 \times 0.8 = 0.118$

$P(S_2 = 10 \mid A_1 = 1, O_1 = w, A_2 = 1) = 0.03 \times 0.1 = 0.003$

- Suppose that after its second action ($A_2 = 1$), it perceives a wall ($O_2 = w$):

$P(S_2 = 6 \mid A_1 = 1, O_1 = w, A_2 = 1, O_2 = w) = 0.003 \times 0.2 = 0.0006 = 0,003$

$P(S_2 = 7 \mid A_1 = 1, O_1 = w, A_2 = 1, O_2 = w) = 0.118 \times 0.9 = 0.1062 = 0,29$

$P(S_2 = 8 \mid A_1 = 1, O_1 = w, A_2 = 1, O_2 = w) = 0.758 \times 0.2 = 0.1516 = 0,41$

$P(S_2 = 9 \mid A_1 = 1, O_1 = w, A_2 = 1, O_2 = w) = 0.118 \times 0.9 = 0.1062 = 0,29$

$P(S_2 = 10 \mid A_1 = 1, O_1 = w, A_2 = 1, O_2 = w) = 0.003 \times 0.9 = 0.0027 = 0,007$

- O_2 doesn't “confirm” A_2

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The localization problem (formalization)

- When the mobile robot is located in front of a door, it perceives:
  - a door with a probability of 80%
  - a wall with a probability of 20%

For $s = \{6, 8, 14\}$

$$P(O_t = d \mid S_t = s) = 0.8$$
$$P(O_t = w \mid S_t = s) = 0.2$$

- When the mobile robot is located in front of a wall, it perceives:
  - a wall with a probability of 90%
  - a door with a probability of 10%

For $s \neq \{6, 8, 14\}$

$$P(O_t = d \mid S_t = s) = 0.1$$
$$P(O_t = w \mid S_t = s) = 0.9$$

- This model is called the sensor model

The localization problem

- We use the sensor model to confirm (or not) the prediction done after the action:

For s in S

$P(S_t = s \mid A_1, O_1, \ldots, A_t, O_t)$

$= P(S_t = s \mid A_1, O_1, \ldots, A_t) \times P(O_t \mid S_t = s)$

The localization problem

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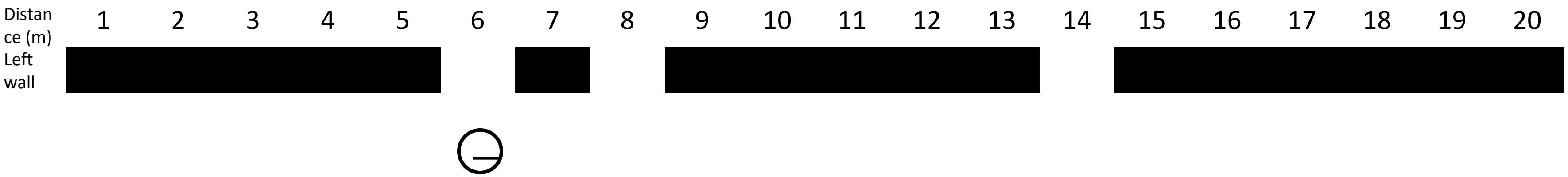


- If the initial position is unknown:

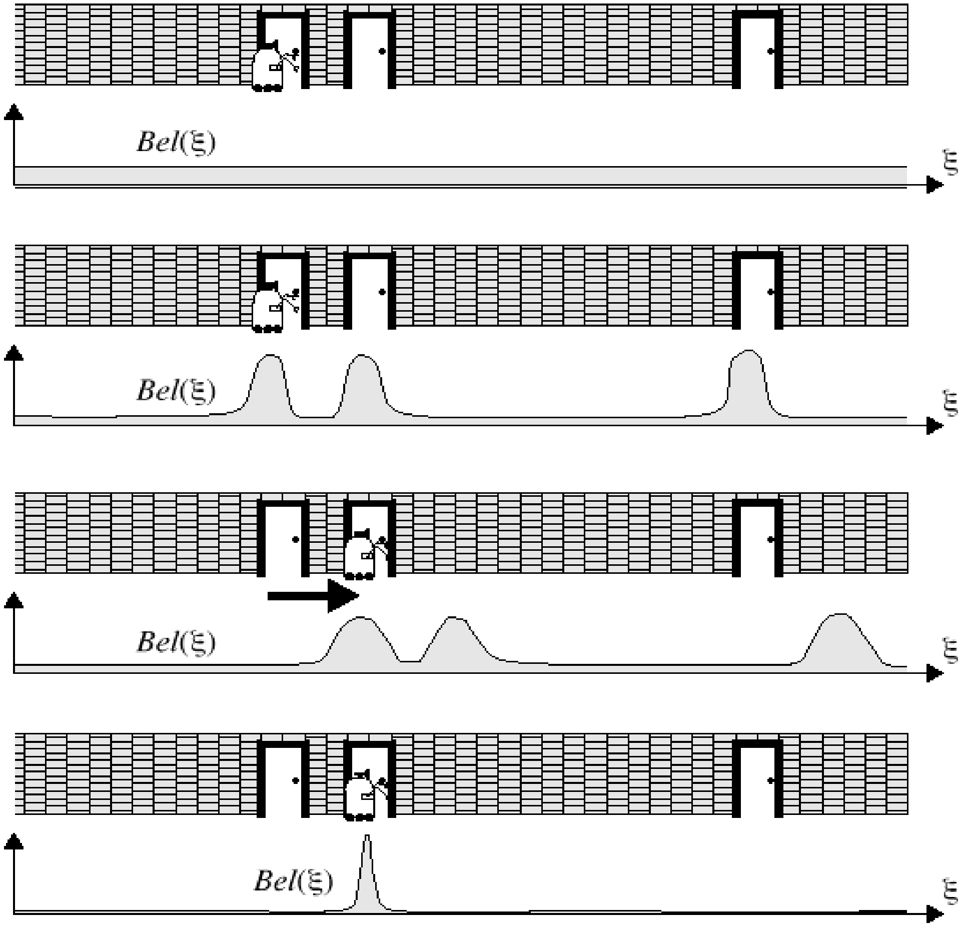
  $P(S_0 = i) = 0.05$ for $i = 1$ to 20

- The mobile robot will make an observation before moving, it perceives a door ($O_0 = d$):

  $P(S_0 = i \mid O_0 = d) = 0.05 \times 0.8 = 0.19$ for $i = 6, 8$ or 14

  $P(S_0 = i \mid O_0 = d) = 0.05 \times 0.1 = 0.025$ for $i \neq 6, 8$ or 14

The localization problem



•[Fox'98]

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Initialization:

$P(S_0|O_0) = \frac{1}{Z} P(S_0) \times P(O_0|S_0)$

Input: $P(S_{T-1}|O_{0:T-1}, A_{1:T-1})$ (previous probability distribution), A_T, O_T

for all $s \in S_T$

$$P(S_T = s|O_{0:T-1}, A_{1:T}) = \sum_{S_{T-1}} P(S_T = s|S_{T-1}, A_T) \times P(S_{T-1}|O_{0:T-1}, A_{1:T-1}) \text{ (prediction)}$$

for all $s \in S_T$

$$P(S_T = s|O_{0:T}, A_{1:T}) = \alpha' P(O_T|S_T = s) \times P(S_T = s|O_{0:T-1}, A_{1:T}) \text{ (estimation : confrontation prediction - observation)}$$

Endfor

return $P(S_T|O_{0:T}, A_{1:T})$

$P(S_T = s|S_{T-1}, A_T)$ is known as the dynamic model and model the uncertainty associated with actions.

$P(O_T|S_T)$ is known as the sensor model and model the uncertainty associated with sensors.

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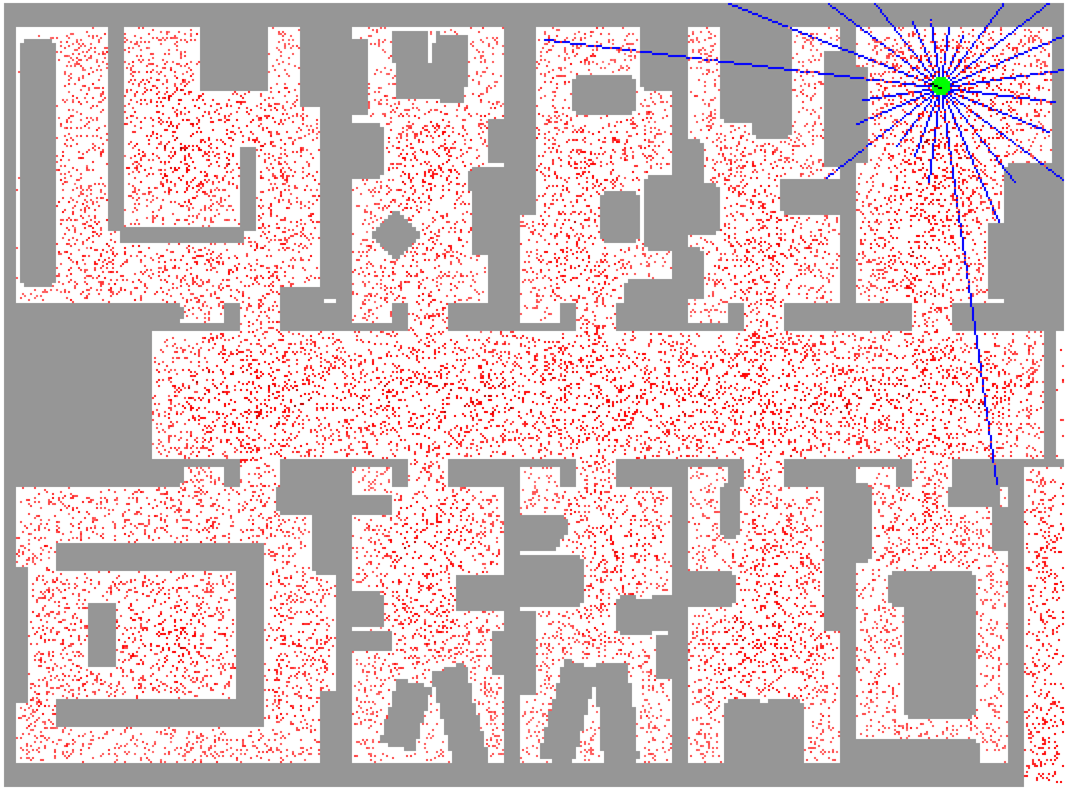
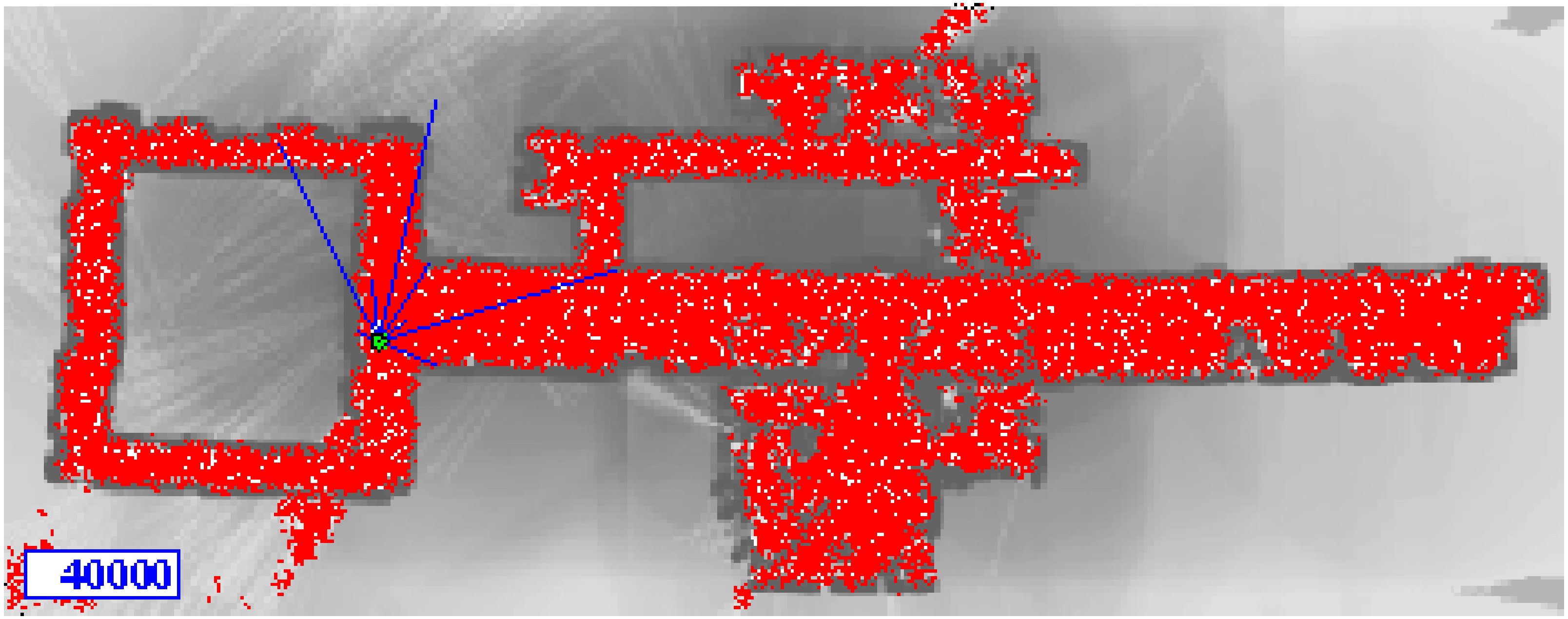
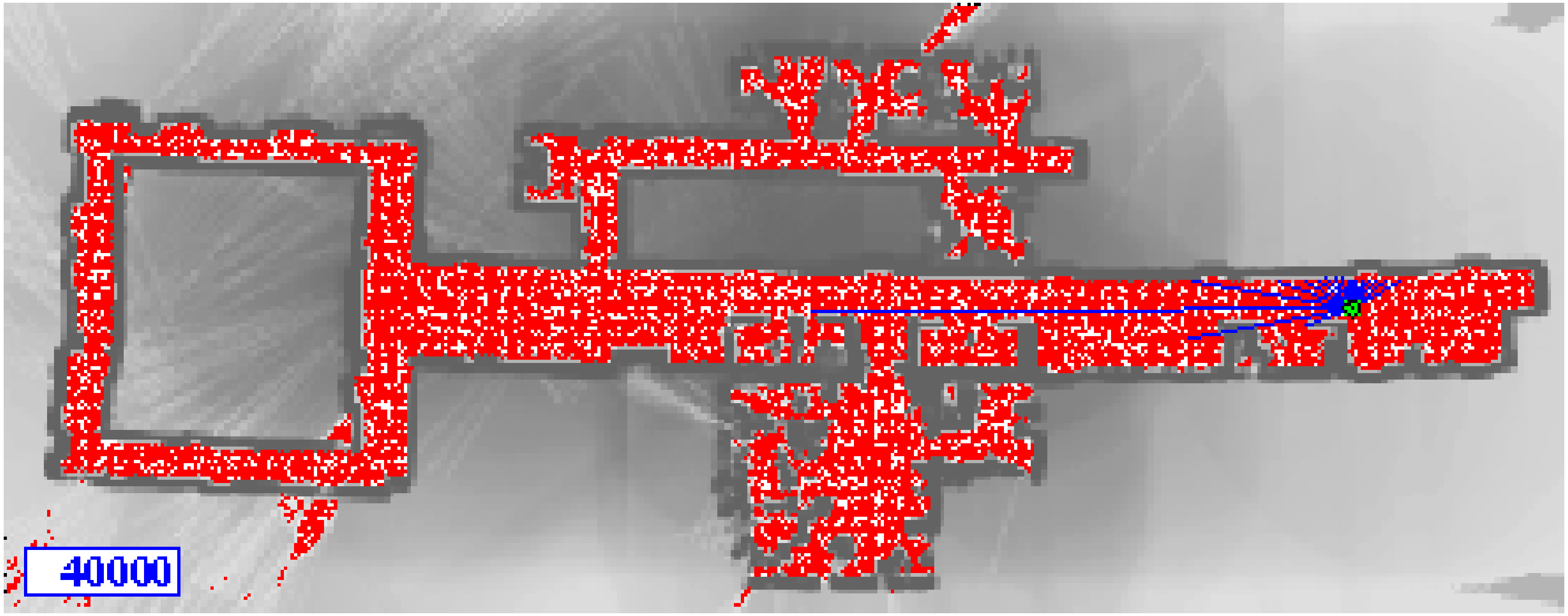
Initial position unknown: ultrasonic sensor[Fox'98]



Initial position unknown: ultrasonic sensor[Fox'98]



Initial position unknown: laser sensor[Fox'98]



The localization problem: conclusion

- The localization algorithm works in 2 steps:
  - Predict of position using actions, actions model and previous localization;
  - Estimation of position using observations, sensor model and prediction;
- The actions model and the sensor model are needed to perform localization: these 2 models are built using knowledge on the mobile robot;
- The localization algorithm works in the same way if initial position is known or unknown: only initial distribution over the state is different;
- There are several implementations of this algorithm:
  - Discrete Bayesian filter;
  - Kalman filter;
  - Particle filter.